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Reducible pattern k-potent ray pattern matrices

Jeffrey Stuart

Department of Mathematics, Pacific Lutheran University, Tacoma, WA 98477-0003, USA

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Abstract

A ray pattern A is called pattern k-potent if k is the smallest positive integer such that $A^{k+1} = A$ as ray patterns. The structure of irreducible, pattern k-potent ray patterns was characterized by Stuart, Beasley and Shader. We extend those results to the reducible case, providing necessary conditions that characterize the structure of each off-diagonal block of the Frobenius normal form of a reducible, pattern k-potent ray pattern.

Keywords: Ray pattern matrix; Pattern k-potent; Reducible ray pattern; Powerful ray pattern

1. Introduction

In this paper, we will be concerned with ray patterns and generalized ray patterns. A *ray pattern* is the set of all complex matrices with a specified zero–nonzero pattern for which only the arguments (modulo 2π) of the nonzero complex entries are specified. A *generalized ray pattern* is obtained from a ray pattern by allowing one or more specified entries to have an ambiguous argument. Such entries are denoted by the symbol #, and the arithmetic properties of # are given in [4]. In particular, the sum of two or more distinct rays results in #. As is customary in the discussion of sign patterns, we will identify a ray pattern with its canonical representative, the matrix whose nonzero entries have modulus 1. That is, a ray pattern or generalized ray pattern will be identified with its element whose entries are from $\mathscr{S} = \{z \in \mathbb{C} : |z| \in \{0, 1\}\}$ or $\mathscr{S} \cup \{\#\}$, respectively. Note that the set $\mathscr{S}$ fills the role played by the set $\{0, +, -\}$ in the study of sign patterns. Also the argument of each nonzero $a \in S$ is unique modulo 2π, and

E-mail address: jeffrey.stuart@plu.edu (J. Stuart).

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hence, we will adopt this uniqueness convention throughout the paper. A pattern B is called a *subpattern* of a pattern A if either B equals A or else B can be obtained from A by replacing one or more nonzero entries with a zero. Two special patterns will be used extensively: I_n, the $n \times n$ identity, and J, the $m \times n$ matrix all of whose entries are 1's.

A real sign pattern A is called *sign k-potent*, if k is the smallest positive integer such that $A^{k+1} = A$ as a sign pattern. A complex ray pattern A is called *pattern k-potent* if k is the smallest positive integer such that $A^{k+1} = A$ as a ray pattern.

Since signature similarity (similarity by a unitary, diagonal matrix) and permutation similarity map ray patterns to ray patterns, two ray patterns will be called *isomorphic* if one can be transformed into the other via signature and/or permutation similarities. If two ray patterns are not isomorphic, then they are called *nonisomorphic*. It is easy to check that signature and permutation similarities commute with the formation of powers, and hence, preserve pattern k-potence.

In a recent paper, [4], Stuart et al. investigated irreducible, pattern k-potent ray patterns, proved a characterization theorem, and introduced a canonical form for such ray patterns. We will use that canonical form to show that many of the results for reducible, sign k-potent sign patterns developed by Stuart [3] have natural extensions to reducible, pattern k-potent ray patterns. In particular, for reducible ray patterns in a special Frobenius normal form, we will determine necessary conditions for pattern k-potence based on the structure of each of the off-diagonal blocks. Throughout this paper we will adopt the notation and conventions of [4]. Note that pattern k-potent ray patterns are closely related to powerful ray patterns studied by Li et al. [2].

In Section 2, the results for the irreducible case are reviewed. In Section 3, it is shown that the rows and columns that correspond to isolated vertices in the directed graph of A play no role in pattern k-potence. Also the cyclic normal form is introduced. Section 4 is devoted to a reduction of order result, Theorem 7, that uses the cyclic normal form. Section 5 contains necessity results characterizing the structure of each of the off-diagonal blocks of a ray pattern A in cyclic normal form satisfying $A^{k+1} = A$. Section 6 contains Theorem 13, which reduces the number of block interrelationships that need to be examined in determining when a ray pattern in cyclic normal form is pattern k-potent.

2. Reduction of order and canonical forms for irreducible ray patterns—red(B) and wP

Suppose that A is a square, irreducible ray pattern. It is well known (see [1, Section 3.4], for example) that there is a unique largest, positive integer m, called the *index of imprimitivity of* A, such that A is permutation similar to an $m \times m$ block partitioned, ray pattern of the form:

$$\hat{A} = \begin{bmatrix} 0 & A_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & A_2 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & 0 & A_{m-1} \\ A_m & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad (*)$$

where the diagonal blocks are square. Further, $\hat{A}$ is unique up to permutation within the blocks and up to cyclic permutation of the sequence of the blocks. The matrix $\hat{A}$ given by $(*)$ is called the *cyclic form of* A. When $m = 1$, A is its own cyclic form, and it will be understood that $A = \hat{A} = A_1$.

It should be apparent that if the irreducible, ray pattern A is in cyclic form with index of imprimitivity m, and that if A is pattern k-potent, then m divides k.

Let P_n denote the $n \times n$ circulant permutation matrix with first row $(0, 1, 0, \dots, 0)$. Let w be a primitive hth root of unity for some positive integer h. (By convention, $w = 1$ is a primitive first root of unity.) Then wP_n is invertible as a ray pattern, with inverse ray pattern $(wP_n)^* = \bar{w}P_n^{\mathrm{T}}$. Also note that n is the smallest positive integer such that $P_n^n = I_n$, implying that $(wP_n)^s = I_n$ whenever s is divisible by both h and n. In particular, $\ell = \operatorname{lcm}(h, n)$ is the smallest such positive integer. Thus, wP_n is pattern ℓ-potent.

Every square, generalized ray pattern admits a symmetric block partition such that each block has the form αJ where $\alpha \in S \cup \{\#\}$ and J is the all ones matrix of the appropriate size. A coarsest block partition of this type is one that is not a proper subpartition of any other symmetric block partition with blocks of the form αJ. Lemma 9 of [5] states that the coarsest block partition of this type always exists and is unique for generalized sign patterns, and that result immediately extends to the case of generalized ray patterns by replacing the symbol set $\{+, -, 0, \#\}$ with $\mathscr{S} \cup \{\#\}$ in the proof given in [5]. For a square, generalized ray pattern A, the *reduced block matrix for* A, denoted $\operatorname{red}(A)$, is the unique ray pattern induced by the coarsest partitioning of A. That is, if the blocks of A in a coarsest partition are $\alpha_{hj}J_{hj}$ for $1 \leqslant h, j \leqslant m$ for some m, then $\operatorname{red}(A)$ is the $m \times m$ ray pattern whose (h, j)-entry is α_{hj}. Observe that wP_n for $w \in \mathscr{S}$ is its own unique, reduced block matrix.

The following result, which is Theorem 12 of [4], follows directly from the proof of Theorem 10 of [5] upon replacing the symbol set $\{+, -, 0, \#\}$ with $\mathscr{S} \cup \{\#\}$, and links reduced block matrices and matrix powers.

Theorem 1. *Let A be a square, generalized ray pattern. Then for each positive integer k,* $\operatorname{red}(A^k) = \operatorname{red}([\operatorname{red}(A)]^k)$. *Further, if $A^{k+1} = A$ for some positive integer k, then* $[\operatorname{red}(A)]^{k+1} = \operatorname{red}(A)$.

The reduced block form plays a central role in the analysis of the irreducible pattern k-potent ray patterns. The following result is Theorem 13 of [4], which is the ray pattern analog of Theorem 11 of [5] and of Theorem 3 of [3].

Theorem 2. *Let A be a nontrivial, irreducible ray pattern. Let k be a positive integer. Then A is pattern k-potent if and only if A can be transformed via signature similarity and permutation similarity into a ray pattern B such that* $\mathrm{red}(B)$ *is wP_m where m is some positive integer that divides k, and where w is a complex number such that w^m is a primitive* (k/m)th *root of unity.*

Thus the collection of ray patterns wP_m where w is a complex number such that w^m is a primitive h-root of unity and h and m are positive integers contains a subcollection of canonical ray patterns for the irreducible, pattern mh-potent ray patterns. As Example 14 of [4] illustrates, within this collection of ray patterns, there is a certain amount of redundancy. Notice that these canonical forms for ray patterns differ from Q_m, which was used as a canonical sign pattern form in [3,5]. In fact, Q_m is *complex* signature similar to $\exp(\mathrm{i}\frac{2\pi}{2m})P_m$, and it is possible to build an alternative, complete set of canonical ray patterns using P_m in conjunction with the matrices $P_m(\frac{2\pi}{r})$ that are obtained by replacing the 1 in the $(m, 1)$-entry of P_m with $\exp(\mathrm{i}\frac{2\pi}{r})$ where r is chosen from the set of positive integers. The following result, which is Proposition 16 of [4], determines which ray patterns are needed to obtain a nonredundant set of canonical ray patterns.

Proposition 3. *Let h and m be positive integers. Let $k = hm$. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ be the distinct positive integers less than or equal to h that are relatively prime to h. Then there are exactly n nonisomorphic, irreducible, pattern k-potent $m \times m$ ray patterns of the form λP_m. These are given by* $\exp(\mathrm{i}\frac{2\pi\gamma_\alpha}{k})P_m$ *for $1 \leqslant \alpha \leqslant n$.*

3. Cyclic normal form

The rest of this paper discusses the powers of reducible ray patterns.

Suppose that A is permutation similar to

$$\begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & B \end{bmatrix}.$$

Note that $A^{k+1} = A$ if and only if $B^{k+1} = B$. Consequently, we will ignore the zero rows and zero columns having a common index set. Such rows and columns will be called *extraneous zero rows and columns*. Note that such rows and columns correspond to the isolated, loop-free vertices in the directed graph $\mathscr{D}$ that has as its adjacency matrix the matrix obtained from A by replacing each nonzero entry with a 1. Henceforth, we will assume that all ray patterns under consideration have had all extraneous zero rows and columns removed.

If the ray pattern A is block upper triangular and pattern k-potent, then necessarily each of the diagonal blocks A_{jj} must satisfy $A_{jj}^{k+1} = A_{jj}$. Then by Lemma 9 of [4], for each j, there exists k_j dividing k such that A_{jj} is pattern k_j-potent, and hence k must be divisible by the least common multiple of the k_j.

Suppose that the ray pattern A has a Frobenius normal form without extraneous zero rows and columns, and with n irreducible, diagonal blocks, A_{jj} such that each of the A_{jj} is pattern k_j-potent for some positive integer k_j. If A_{jj} is the 1×1 zero matrix, use the 1×1 pattern I_1 as a signature and permutation matrix. If A_{jj} is not a zero matrix, then apply Theorem 2 to obtain permutation and signature matrices such that A_{jj} is isomorphic to a ray pattern $\hat{A}_{jj}$ such that $\operatorname{red}(\hat{A}_{jj})$ is of the form $w_j P_{m_j}$ for some positive integer m_j and some complex number w_j such that $w_j^{m_j}$ is a primitive (k_j/m_j)th root of unity. Using the direct sum of the permutation similarities for the diagonal blocks and the direct sum of the signature similarities for the diagonal blocks, A is isomorphic to a ray pattern $\hat{A}$ in Frobenius normal form whose irreducible, diagonal blocks are either 1×1 zero matrices or else the $\hat{A}_{jj}$. Any matrix $\hat{A}$ in this form is called the *cyclic normal form for* A.

Example 4. Let A be the following partially specified, 7×7 ray pattern. The solid lines indicate the partition for the Frobenius normal form, and the dashed lines indicate a subblock partition induced by the subblock partition of the cyclic subblocks of the irreducible blocks.

$$A = \left[\begin{array}{cc:c|cc|c|c}
0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & \multicolumn{2}{c|}{A_{13}} & A_{14} & A_{15} \\
0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & & & & \\
\hdashline
\exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & 0 & \multicolumn{2}{c|}{A_{23}} & A_{24} & A_{25} \\
\hline
0 & 0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & A_{34} & A_{35} \\
0 & 0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{45} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Note that A is in cyclic normal form with four irreducible diagonal blocks. The first irreducible diagonal block is either pattern $2h_1$-potent (h_1 odd) or pattern h_1-potent (h_1 even), the second is pattern h_2-potent, and the remaining two diagonal blocks are pattern 1-potent. If A is pattern k-potent, then k is divisible by $\operatorname{lcm}\{2, h_1, h_2\}$.

4. The role of red(A) for reducible ray patterns

We next show that it is not the partition induced by the Frobenius normal form of A but rather the subpartition induced by the cyclic form of each irreducible, diagonal block of A that is crucial to understanding the structure of A when A is pattern k-potent. These results parallel the results developed for the reducible sign patterns in [3].

Lemma 5. *Let A be a ray pattern in cyclic normal form. Let A be subblock partitioned by the partition induced by the cyclic blocks of each irreducible diagonal block of the Frobenius normal form. If $A^{k+1} = A$ for some $k \geqslant 1$, then for all $h < j$, the subblock A_{hj} is given by $A_{hj} = \alpha_{hj} J_{hj}$ where $\alpha_{hj} \in \mathscr{S}$ and J_{hj} is the all ones matrix the same size as A_{hj}.*

Proof. This lemma is the ray pattern analog of Lemma 7 of [3]. The proof follows directly from the proof of Lemma 7 of [3] upon replacing the symbol set $\{0, +, -\}$ with the set $\mathscr{S}$, and upon making the following substitutions: Substitute Lemma 5 of [4] for Lemma 6 of [5], and substitute the remark immediately preceding Example 4 in this paper for Theorem 4 of [5]. $\square$

Example 6. Let A be the partially specified ray pattern in Example 4. For $h < j$, let J_{hj} be the all ones matrix that is the same size as A_{hj}. If $A^{k+1} = A$ for some k, then $\mathrm{lcm}\{2, h_1, h_2\}$ divides k, and A must have the following structure:

$$A = \left[\begin{array}{cc:c|cc|c|c}
0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & \multicolumn{2}{c|}{\multirow{2}{*}{$\alpha_{13}J_{13}$}} & \multirow{2}{*}{$\alpha_{14}J_{14}$} & \multirow{2}{*}{$\alpha_{15}J_{15}$} \\
0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & & & & \\
\hdashline
\exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & 0 & \multicolumn{2}{c|}{\alpha_{23}J_{23}} & \alpha_{24}J_{24} & \alpha_{25}J_{25} \\
\hline
0 & 0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \multirow{2}{*}{$\alpha_{34}J_{34}$} & \multirow{2}{*}{$\alpha_{35}J_{35}$} \\
0 & 0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \alpha_{45}J_{45} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

where the α_{hj} are in S. Since a product of the form $J_{hj}J_{jk}$ will again be an all ones matrix, A will be pattern k-potent if and only if the smaller matrix $B = \mathrm{red}(A)$ is pattern k-potent:

$$B = \left[\begin{array}{ccc|c|c}
0 & \exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & \alpha_{13} & \alpha_{14} & \alpha_{15} \\
\exp\left(\mathrm{i}\frac{2\pi}{h_1}\right) & 0 & \alpha_{23} & \alpha_{24} & \alpha_{25} \\
0 & 0 & \exp\left(\mathrm{i}\frac{2\pi}{h_2}\right) & \alpha_{34} & \alpha_{35} \\
\hline
0 & 0 & 0 & 0 & \alpha_{45} \\
\hline
0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Note that B is in cyclic normal form with diagonal blocks of type $\exp(\mathrm{i}\frac{2\pi}{h})P$ and $[0]$.

Now we are ready to state a reduction theorem for reducible, pattern k-potent matrices.

Theorem 7. *Let A be a ray pattern in cyclic normal form with no extraneous zero rows and columns. Let A be subblock partitioned by the partition induced by the cyclic blocks of each irreducible, diagonal block of the cyclic normal form. The following are equivalent*:

(i) *The matrix* $A^{k+1} = A$ *for some positive integer* k;
(ii) *For all* h *and* j, *the subblock* $A_{hj} = \alpha_{hj} J_{hj}$ *where* $\alpha_{hj} \in \mathscr{S}$ *and* J_{hj} *is the all ones matrix the same size as* A_{hj}; *and the matrix of coefficients*, $\operatorname{red}(A)$, *satisfies* $[\operatorname{red}(A)]^{k+1} = \operatorname{red}(A)$ *for some positive integer* k.

Proof. (i) $\Rightarrow$ (ii). By Lemma 5, every subblock of A is of the form αJ. Since the irreducible, diagonal blocks are in cyclic form, and since A has no extraneous zero rows and columns, the subblock partition that they induce on A is the coarsest possible symmetric partition of A into blocks of the form αJ. Thus the matrix of subblock coefficients is $\operatorname{red}(A)$. Now apply Theorem 1.

(ii) $\Rightarrow$ (i). The cyclic structure of the diagonal blocks forces A and A^{k+1} to have the same coarsest, symmetric block partitioning into blocks with constant argument. Thus $\operatorname{red}(A^{k+1}) = [\operatorname{red}(A)]^{k+1} = \operatorname{red}(A)$, together with $J_{r\times s} J_{s\times t} = J_{r\times t}$ for all positive integers r, s and t, implies $A^{k+1} = A$. $\square$

One important consequence of Theorem 7 is that the problem of classifying which ray patterns are pattern k-potent reduces to examining those patterns in cyclic normal form for which the irreducible, diagonal blocks are 1×1 zero matrices and matrices of the type wP.

5. The structure of A_{rs} for $s > r$

The next lemma will be used to derive a set of necessary conditions on the structure of the individual off-diagonal blocks of a pattern k-potent matrix A.

Lemma 8. *Let* A *be a block upper triangular ray pattern with diagonal blocks* A_{jj}. *Suppose* $s > r$. *If* $A^{k+1} = A$ *for some positive integer* k, *then* $(A_{rr})^{k-h} A_{rs} (A_{ss})^h$ *is a subpattern of* A_{rs} *for* $0 \leqslant h \leqslant k$. *Further, if* A_{rr} *is of type* $w_r P_{m_r}$ *and if* A_{ss} *is of type* $w_s P_{m_s}$, *then* $A_{rr} A_{rs} = A_{rs} A_{ss}$.

Proof. For any block upper triangular matrix, the (r, s)-block of A^{k+1} contains among its summands

$$\sum_{h=0}^{k} (A_{rr})^{k-h} A_{rs} (A_{ss})^h. \qquad (**)$$

If $A^{k+1} = A$, then each summand must be a subpattern of the ray pattern A_{rs}.

Now suppose that A_{rr} and A_{ss} are of type wP. Since patterns of type wP are invertible as ray patterns, no annihilation of nonzero entries can occur in products involving such matrices, so $(A_{rr})^p A_{rs} (A_{ss})^q$ has the same number of nonzero entries for all nonnegative integers p and q. But since $A_{rr} A_{rs} (A_{ss})^{k-1}$ and $A_{rs}(A_{ss})^k$ are both subpatterns of A_{rs}, it follows that $A_{rr} A_{rs} (A_{ss})^{k-1} = A_{rs} = A_{rs} (A_{ss})^k$.

Multiplying $A_{rr}A_{rs}(A_{ss})^{k-1} = A_{rs}$ on the right by A_{ss} yields $A_{rr}A_{rs}(A_{ss})^k = A_{rs}A_{ss}$. Now substitute $A_{rs} = A_{rs}(A_{ss})^k$. □

5.1. *Case I: At least one of A_{rr} and A_{ss} is [0]*

Note that the (r, s)-block of A^{k+1} contains summands in addition to those in $(**)$ exactly when $s > r + 1$. Suppose that $A_{rr} = [0]$, the 1×1 zero block. Then all summands $(A_{rr})^{k-h}A_{rs}(A_{ss})^h$ in $(**)$ are zero vectors, except possibly when $h = k$. If A_{ss} is of type wP_m, then $(A_{ss})^{k+1} = A_{ss}$ implies $(A_{ss})^k = I_m$, and hence $(**)$ imposes no restrictions on A_{rs}. Arguing analogously when $A_{ss} = [0]$ yields:

Theorem 9. *Let A be a block upper triangular ray pattern with diagonal blocks A_{jj}. Let s and r be positive integers with $s > r$. Suppose $A^{k+1} = A$ for some positive integer k. If exactly one of A_{rr} and A_{ss} is equal to the 1×1 zero matrix, and the other is of type wP, then $A_{rr}A_{rs} = A_{rs}A_{ss}$ need not hold. If both of A_{rr} and A_{ss} are equal to the 1×1 zero matrix, then $A_{rr}A_{rs} = A_{rs}A_{ss}$ trivially holds. If at least one of A_{rr} and A_{ss} is equal to the 1×1 zero matrix and if $s > r + 1$, then $(**)$ imposes no restrictions on the entries of the ray pattern vector A_{rs}. If both of A_{rr} and A_{ss} are equal to the 1×1 zero matrix and if $s = r + 1$, then A_{rs} must be the 1×1 zero matrix.*

Example 10. Let A be the pattern 1-potent ray pattern

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A_{11} = A_{33} = [0]$, both of the summands in $(**)$ are [0], but $A_{13} = [1]$.

5.2. *Case II: A_{rr} and A_{ss} are of type wP*

For convenience, let $B = A_{rs}$. Suppose that $A_{rr} = w_1P_m$ and that $A_{ss} = w_2P_n$, where w_1 and w_2 are nonzero, and that further, $A_{rr}B = BA_{ss}$.

First consider the case when $w_1 = w_2 = w$ and A_{rr} and A_{ss} are the same size: $n \times n$. Then $A_{rr}B = BA_{ss}$ is equivalent to $P_nB = BP_n$, and as is well known, this is equivalent to the condition that B is a circulant ray pattern. That is, $B = \sum_{h=0}^{n-1} b_hP_n^h$ where each b_h is chosen independently from $\mathscr{S}$.

Next, keep the assumption that $w_1 = w_2$, and suppose that $m \neq n$. Then $P_mB = BP_n$ still holds, but the structure of B is more complicated. In [6], it is shown that for $m \neq n$, the ray pattern B must decompose into copies of a smaller, $g \times g$ circulant ray pattern C where $g = \gcd\{m, n\}$. Further, B is banded with mn/g identical elements corresponding to each independently chosen entry in the first row of C, and that B is block partitioned into m/g block rows and n/g block columns, with each subblock given by C. The effect of the permutations is to push elements in B along a diagonal band until they reach an edge of the matrix, and then to wrap them back onto the continuation of the diagonal band using row and column indexes modulo m and modulo n, respectively.

Now allow that $w_1 \neq w_2$. Let $\rho = w_2/w_1$. Then $A_{rr}B = BA_{ss}$ is equivalent to $B = \rho P_m^{-1} B P_n$. Again following the structural analysis given in [6] for matrices B satisfying $P_m B = B P_n$, each entry of B lies on exactly one of $g = \gcd\{m, n\}$ diagonal "bands" that wrap around B. These g bands are given by the index sets $\{(\alpha, \beta) : \alpha - \beta \equiv h \bmod g\}$ for $1 \leqslant h \leqslant g$. Further, the effect of the permutation similarity is that for each choice of indices α and β, $B_{\alpha+1,\beta+1} = \rho B_{\alpha\beta}$, where the first index is read $\bmod m$ and the second index is read $\bmod n$. Starting at $B_{1\beta}$ for some β with $1 \leqslant \beta \leqslant g$, traveling down the band, wrapping back around when either the bottom or right-hand side of B is reached, we return to $B_{1\alpha}$ after mn/g steps. Thus $B_{1\beta} = \rho^{mn/g} B_{1\beta}$. Hence either B is the zero matrix, or else, $\rho^{mn/g} = 1$.

Note that the $g \times g$ blocks of B themselves fall naturally into bands as discussed in [6]. Starting at any entry $B_{\alpha\beta}$, follow along the diagonal "band". After g entries, the next g entries are ρ^g times the first g entries. Thus, if we follow the $g \times g$ block in the upper left-hand corner of B, call it C, we see that the next $g \times g$ block along the block "band" is $\rho^g C$. Now B is block partitioned into m/g rows of blocks and n/g columns of blocks. Since m/g and n/g are relatively prime by the choice of g, the block "band" containing block C actually winds through every one of the partition blocks of B. Thus each of the natural $g \times g$ subblocks of B is of the form $\rho^{gh} C$ for some nonnegative integer h. Further, given indices μ and ν with $1 \leqslant \mu \leqslant m/g$ and $1 \leqslant \nu \leqslant n/g$, there exists a smallest nonnegative integer h such that $\mu \equiv 1 + h \bmod (m/g)$ and $\nu \equiv 1 + h \bmod (n/g)$. Then the (μ, ν) block of B is given by $\rho^{gh} C$ for that value of h.

Finally let us examine the structure of C carefully. If the first row of C is given by $R = [b_0, b_1, \ldots, b_{g-1}]$, then for $1 \leqslant \alpha \leqslant \beta \leqslant g$, $C_{\alpha\beta} = \rho^{\alpha-1} b_{\beta-\alpha}$. If $D = \operatorname{diag}(1, \rho, \rho^2, \ldots, \rho^{g-1})$, then C can be expressed as $C = DM$ where M has the same entries on and above the diagonal as the circulant matrix with first row R. It remains to determine the entries of M that lie below the diagonal. The effect of the permutation similarity on the rows of B is to shift them down and over to the right, and at each such shift, to multiply the row by ρ. Since g divides both m and n, it follows that the section of each row of C that is to the left of the diagonal is ρ to some row-dependent, positive power multiplying a corresponding section of a cyclic permutation of R. In particular, since $C_{gg} = \rho^{g-1} b_0$, it follows that the section of the last row of C that is to the left of the diagonal must be $\rho^t [b_1, b_2, \ldots, b_{g-1}]$ for some positive integer t. Specifically, let t be the smallest positive integer such that applying t shifts and scalings to the first row of B carries $B_{12} = b_1$ to $B_{g1} = \rho^t b_1$. Note that $t \geqslant g - 1$, and that $1 + t \equiv g \bmod m$ and $2 + t \equiv 1 \bmod n$. Starting with the last row of C, and tracing backwards k rows, the section of the $(g - k)$th row of C that is to the left of the diagonal must be $\rho^{t-k} [b_{k+1}, b_{k+2}, \ldots, b_{g-1}]$. Thus for $1 \leqslant \beta < \alpha \leqslant g$,

$$
\begin{aligned}
C_{\alpha\beta} &= \rho^{t-g+\alpha} b_{g-(\alpha-\beta)} \\
&= \rho^{\alpha-1}\rho^{t-g+1} b_{g-(\alpha-\beta)} \\
M_{\alpha\beta} &= \rho^{t-g+1} b_{g-(\alpha-\beta)}.
\end{aligned}
$$

For convenience, let $\xi = t - g + 1$. Then the conditions on t are equivalent to ξ is the smallest nonnegative integer such that $\xi \equiv 0 \bmod m$ and $\xi + g \equiv 0 \bmod n$. Notice that the entries below the diagonal of M are ρ^{ξ} times the corresponding entries of the circulant matrix with first row R.

Thus we have proven:

Theorem 11. *Let $A_{rr} = w_1 P_m$ and $A_{ss} = w_2 P_n$ with w_1 and w_2 in $\mathscr{S} \setminus \{0\}$. Let $\rho = w_2/w_1$ and let $g = \gcd\{m, n\}$. Let ξ be the smallest nonnegative integer satisfying both $\xi \equiv 0 \bmod m$ and $\xi + g \equiv 0 \bmod n$. If $\rho^{mn/g} \neq 1$, then $A_{rr}A_{rs} = A_{rs}A_{ss}$ if and only if $A_{rs} = 0$. If $\rho^{mn/g} = 1$, then $A_{rr}A_{rs} = A_{rs}A_{ss}$ if and only if A_{rs} decomposes as copies of the $g \times g$ scaled circulant-like pattern matrix $C = D\sum_{j=0}^{g-1} b_j Q_j$ where $D = \mathrm{diag}(1, \rho, \rho^2, \ldots, \rho^{g-1})$, where Q_j is obtained from P_m^j by replacing each of the ones below the main diagonal with ρ^{ξ}, and where b is an arbitrary $g \times 1$ ray pattern vector. For $1 \leqslant \mu \leqslant m/g$ and $1 \leqslant \nu \leqslant n/g$, the (μ, ν)-block of A_{rs} is given by $\rho^{gh}C$ where h is the smallest nonnegative integer such that $\mu \equiv 1 + h \bmod (m/g)$ and $\nu \equiv 1 + h \bmod (n/g)$.*

It is worth noting that the existence of a positive integer ξ satisfying both $\xi \equiv 0 \bmod m$ and $\xi + g \equiv 0 \bmod n$ is equivalent to the condition that m/g and n/g are relatively prime, which is, of course, an immediate consequence of the fact that $g = \gcd\{m, n\}$.

It might be useful to compare this result to Theorem 13 of [3], which was the analogous result for sign patterns. Note that Q_m is complex signature similar to $w_1 P_m$ with $w_1 = \exp(\mathrm{i}\frac{2\pi}{2m})$ and that Q_n is complex signature similar to $w_2 P_n$ with $w_2 = \exp(\mathrm{i}\frac{2\pi}{2n})$. The condition that $(\frac{w_2}{w_1})^{mn/g} = 1$ is equivalent to requiring that $\frac{mn}{g}(\frac{1}{2n} - \frac{1}{2m})$ be an integer. That is, $(m - n)/g$ is an even integer, or equivalently, $(m + n)/g$ is an even integer, precisely the condition given in Theorem 13 of [3]. The structure given here for A_{rs} differs somewhat from that given in [3] because we have used an alternative canonical form for A_{rr} and A_{ss}.

6. More necessary conditions for pattern k-potence

Let A be in cyclic normal form. Suppose $A^{k+1} = A$. Then each A_{rr} is pattern k_r-potent for some k_r that divides k according to Lemma 9 of [4]. Thus k is divisible by $\ell = \mathrm{lcm}\{k_1, k_2, \ldots, k_n\}$. By the theorems in the previous section, the structure of each block A_{rs} with $s > r$ is partially determined. It remains to examine the relationship between the off-diagonal blocks.

Example 12. Suppose that A is in cyclic normal form with three irreducible, diagonal blocks: $w_1 P_3$, $w_2 P_2$ and $w_3 P_2$, where w_1^3 is a primitive (h_1)th root of unity, and where w_2^2 and w_3^2 are primitive (h_2)th and (h_3)th roots of unity, respectively, for some positive integers h_1, h_2 and h_3. Suppose that A is pattern k-potent for some positive integer k. Then k is divisible by $\operatorname{lcm}\{h_1, h_2, h_3, 3, 2, 2\}$. Since $\gcd\{3, 2\} = 1$ and $\gcd\{2, 2\} = 2$, A must be of the form:

$$A = \left[\begin{array}{ccc|cc|cc} 0 & w_1 & 0 & a & \rho_{12}^3 a & b & \rho_{13}^3 b \\ 0 & 0 & w_1 & \rho_{12}^4 a & \rho_{12} a & \rho_{13}^4 b & \rho_{13} b \\ w_1 & 0 & 0 & \rho_{12}^2 a & \rho_{12}^5 a & \rho_{13}^2 b & \rho_{13}^5 b \\ \hline \multicolumn{3}{c|}{\multirow{2}{*}{$0_{2\times 3}$}} & 0 & w_2 & c & d \\ & & & w_2 & 0 & \rho_{23} d & \rho_{23} c \\ \hline \multicolumn{3}{c|}{\multirow{2}{*}{$0_{2\times 3}$}} & \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & 0 & w_3 \\ & & & & & w_3 & 0 \end{array}\right],$$

where $\rho_{12} = w_2/w_1$, $\rho_{13} = w_3/w_1$ and $\rho_{23} = w_3/w_2$. Further, either $a = 0$ or $\rho_{12}^6 = 1$, $b = 0$ or $\rho_{13}^6 = 1$, and $c = d = 0$ or $\rho_{23}^2 = 1$. Suppose that $\rho_{12}^6 = \rho_{13}^6 = \rho_{23}^2 = 1$. Then $w_1 = \exp(\mathrm{i}\frac{2\pi t}{6})w_2$ for some integer t, and $w_3 = \pm w_2$. Further, the four parameters a, b, c and d cannot be chosen independently from $\mathscr{S}$ because in order for A^2 to be unambiguous, it can be shown using Maple® that they satisfy the additional requirement that $w_1^3(w_3 b + ad) + w_2^2 w_3 ac$ and $w_3^4 b + w_1^3 ac + w_2^2 w_3 ad$ must both be unambiguous. In the particular case where $w_1 = \exp(\mathrm{i}\frac{2\pi}{6})$, $w_2 = 1$ and $w_3 = -1$, it follows that $h_1 = 2$, $h_2 = 1$, $h_3 = 2$, and that k is divisible by 6 (in fact, $k = 6$), and that $b - a(c + d)$ must be unambiguous.

Suppose A is a block upper triangular ray pattern. For $s > r$, $(A^{k+1})_{rs}$ is the sum of all products of the form

$$A_{r_0r_0}^{h_0} A_{r_0r_1} A_{r_1r_1}^{h_1} A_{r_1r_2} A_{r_2r_2}^{h_2} \cdots A_{r_{q-1}r_q} A_{r_qr_q}^{h_q},$$

where q is an integer with $1 \leqslant q \leqslant k+1$, where $r = r_0 < r_1 < \cdots < r_q = s$, where each h_t is a nonnegative integer for $0 \leqslant t \leqslant q$, and where $q + \sum_{t=0}^{q} h_t = k+1$. Suppose that $A_{r_\alpha r_\alpha} A_{r_\alpha r_{\alpha+1}} = A_{r_\alpha r_{\alpha+1}} A_{r_{\alpha+1} r_{\alpha+1}}$ for $0 \leqslant \alpha < q$. Then

$$\begin{aligned} &A_{r_0r_0}^{h_0} A_{r_0r_1} A_{r_1r_1}^{h_1} A_{r_1r_2} A_{r_2r_2}^{h_2} \cdots A_{r_{q-1}r_q} A_{r_qr_q}^{h_q} \\ &\quad = A_{r_0r_0}^{k+1-q} A_{r_0r_1} A_{r_1r_2} A_{r_2r_3} \cdots A_{r_{q-1}r_q}. \end{aligned}$$

Since $A_{r_0r_0}^{h_0} A_{r_0r_1} A_{r_1r_1}^{h_1} A_{r_1r_2} A_{r_2r_2}^{h_2} \cdots A_{r_{q-1}r_q} A_{r_qr_q}^{h_q}$ must be a subpattern of A_{rs} if $A^{k+1} = A$, the following result holds:

Theorem 13. *Let A be in cyclic normal form. Let k be a positive integer. Suppose that every irreducible diagonal block is of type wP such that $(wP)^{k+1} = wP$. If $A_{rr}A_{rs} = A_{rs}A_{ss}$ for all $r < s$; and if for each sequence $r = r_0 < r_1 < \cdots < r_q = s$ with $1 \leqslant q \leqslant k+1$, the generalized ray pattern $A_{rr}^{k+1-q}A_{rr_1}A_{r_1r_2}\cdots A_{r_{q-1}s}$ is an unambiguous subpattern of A_{rs}, then $A^{k+1} = A$.*

This theorem reduces the number of block products that need to be examined for ambiguous entries in testing whether A is pattern k-potent. Note that in Example 12, the restriction that $b - a(c+d)$ is unambiguously patterned is precisely the condition that $A_{11}^{k+1-2}A_{12}A_{23}$ is unambiguously patterned and hence a subpattern of A_{13}.

As noted in Theorem 9, when one or more of the diagonal blocks of A is a 1×1 zero block, it may no longer possible to pass the diagonal blocks through the path product of blocks, and consequently, more block products will need to be examined. Consider the following example from [3]:

Example 14. Let A be the ray pattern

$$A = \begin{bmatrix} 0_{1\times 1} & x^{\mathrm{T}} & 0_{1\times 1} & y^{\mathrm{T}} \\ 0_{m\times 1} & P_m & v & B \\ 0_{1\times 1} & 0_{1\times m} & 0_{1\times 1} & z^{\mathrm{T}} \\ 0_{n\times 1} & 0_{n\times m} & 0_{n\times 1} & P_n \end{bmatrix},$$

where B decomposes as copies of a $g \times g$ circulant ray pattern with $g = \gcd\{m, n\}$, and where v, x, y, z are pattern column vectors. $A^{k+1} = A$ if and only if: k is divisible by $\operatorname{lcm}\{m, n\}$; $x^{\mathrm{T}}P_m^{k-1}v = 0_{1\times 1}$; $x^{\mathrm{T}}BP_n^{k-1}$ is a subpattern of y^{T}; $x^{\mathrm{T}}P_m^h vz^{\mathrm{T}}P_n^{k-h-2}$ is a subpattern of y^{T} for $0 \leqslant h \leqslant k-2$; and $P_m^h vz^{\mathrm{T}}P_n^{k-h-1}$ is a subpattern of B for $0 \leqslant h \leqslant k-1$. Note that $x^{\mathrm{T}}P_m^{k-1}u$ cannot be replaced with a nontrivial condition involving $A_{11}^{k-1}A_{12}A_{23}$ or $A_{12}A_{23}A_{33}^{k-1}$, since these latter products impose no restrictions on x or u. Also note that $P_m^h vz^{\mathrm{T}}P_n^{k-h-1}$ apparently cannot be replaced with a condition involving $A_{22}^{k-1}A_{23}A_{34} = P_m^{k-1}vz^{\mathrm{T}}$ or $A_{23}A_{34}A_{44}^{k-1} = vz^{\mathrm{T}}P_n^{k-1}$. Finally note that $A_{11} = A_{33} = 0_{1\times 1}$ do not require $A_{13} = 0_{1\times 1}$; if A_{13} is chosen to be nonzero, then the set of products that must be examined is even more complicated.

We end this paper with two open questions. Is there a more complete characterization of the reducible case, one that usefully specifies the relationships between the parameters in distinct, off-diagonal blocks of $\operatorname{red}(A)$? And, second, if the cyclic normal form of a pattern k-potent ray pattern has n irreducible, diagonal blocks, and for $1 \leqslant \alpha \leqslant n$, the αth diagonal block is pattern k_α-potent, then is $k = \operatorname{lcm}\{k_1, k_2, \ldots, k_n\}$, or are there examples of such ray patterns for which $\operatorname{lcm}\{k_1, k_2, \ldots, k_n\}$ is a proper divisor of k?

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